

Solving the 2-D Vorticity-streamfunction equations. HW 4: Solving the discretized system of equations

Computational Goal: Evolve the coupled Vorticity-streamfunction equations forward in time for 9 time-steps. The assignment asks for the values of $\omega(32, 32)$ at these time steps.

1 The system of equations?

The governing vorticity-streamfunction equations that describe 2-D advection-diffusion are:

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega - [\psi, \omega] \quad (1)$$

$$\nabla^2 \psi = \omega \quad (2)$$

where,

$$[\psi, \omega] = \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} \quad (3)$$

Equations equation (1) and equation (2) after the discretization in HW 3 take the form:

$$\omega_t = \nu A\psi - B\psi C\omega + C\psi B\omega \quad (4)$$

$$A\psi = \omega \quad (5)$$

This is a system of $N^2 \times 1$ equations.

2 The domain, intial and boundary conditions, and parameter values?

The boundary conditions for both vorticity and streamfunctions are considered to be periodic, and the equations are studied in a square domain $(x, y) \in [-L, L] = [-10, 10]$. Discretize the domain using $N = 64$ grid points in the (x, y) directions with a uniform grid spacing $\delta = \Delta x = \Delta y$. The following code snippet should help in setting up your domain:

```
n = 64; L = 10;
x2 = linspace(-L,L,n+1); % The grid
x = x2(1:n); % Periodic BC, hence consider only n points
del = mean(diff(x)); % del = del_x = del_y = grid spacing
y = x; % Square domain, uniform grid spacing
[X,Y] = meshgrid(x,y); % The 2D grid
```

Take the kinematic viscosity $\nu = 0.001$.

The initial condition is a gaussian:

$$\omega(x, y, 0) = \exp(-x^2 - \frac{y^2}{20}) \quad (6)$$

Solve the equations for $t_{span} = 0 : 0.5 : 4$. To avoid a singular system of equations while solving 5, take $A(1, 1) = 2/\delta^2$ instead of $A(1, 1) = -4/\delta^2$. **NOTE:** You can use the modified matrix A to solve 4 as well.

3 The algorithm steps?

The governing equations describe two-dimensional fluid advection-diffusion behavior of the vorticity of the fluid $\omega(x, y, t)$ coupled with the streamfunction $\psi(x, y, t)$. Thus there are two coupled equations to solve:

1. Starting with vorticity in the previous time step, the poisson equation equation (2) which has second-order spatial derivative terms has to be solved for streamfunction at the next time step
2. The updated streamfunction is used in equation (1), which has both first and second order spatial derivatives, first order time derivative. This equation has to be advanced in time to get the vorticity at the next time step

The spatial derivatives are discretized using second finite-difference schemes following HW 3. The poisson equation 5 is a sparse $Ax = b$ system of equations. Equation 4 is an ODE, and is stepped forward in time using the MATLAB `ode45` algorithm.

4 The implementation details?

Some implementation details you should keep in mind:

1. Reshape the initial vorticity as an $N^2 \times 1$ vector first as per our ordering, using the '`reshape`' command in MATLAB, and work with these reshaped vectors throughout. You can later retrieve the values for $\omega(32, 32)$ at each of the time steps from your solution vector. The solution vector should be $9 \times N^2$, with each row being the solution values for each time step.
2. While defining 4, note that the operations $B\psi$, $C\omega$, $C\psi$, and $B\omega$ will all give $N^2 \times 1$ vectors. Hence to compute $B\psi C\omega$ and $C\psi B\omega$ terms use elementwise operations: '`.*`'. This is consistent with the equation 4.