

Solving the 2-D Vorticity-streamfunction equations. HW 3: Setting up the spatial derivative matrices

Computational Goal: Derive first and second order spatial derivative matrices from second order accurate Finite-difference schemes

The governing vorticity-streamfunction equations that describe 2-D advection-diffusion are:

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega - [\psi, \omega] \quad (1)$$

$$\nabla^2 \psi = \omega \quad (2)$$

where,

$$[\psi, \omega] = \frac{\partial \psi}{\partial x} \frac{\partial \omega}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \omega}{\partial x} \quad (3)$$

The boundary conditions for both vorticity and streamfunctions are considered to be periodic, and the equations are studied in a square domain $(x, y) \in [-L, L]$. ν = Kinematic viscosity.

1 How do I discretize the spatial derivatives?

Use a second order accurate finite difference schemes on a 2-D domain to discretize the spatial derivative terms. Second order centered difference schemes with a uniform grid $\delta = \Delta x = \Delta y$ are given by:

$$\frac{\partial f}{\partial x} \approx \frac{f(x + \Delta x, y) - f(x - \Delta x, y)}{2\delta} \quad (4)$$

$$\frac{\partial f}{\partial y} \approx \frac{f(x, y + \Delta y) - f(x, y - \Delta y)}{2\delta} \quad (5)$$

$$\nabla^2 f \approx \frac{f(x + \Delta x, y) + f(x - \Delta x, y) + f(x, y + \Delta y) - f(x, y - \Delta y) - 4f(x, y)}{\delta^2} \quad (6)$$

2 How do I order my grid points?

Use a lexicographical ordering for this assignment: ordering the grid points such that the 2-D matrix of unknowns $f_{ij} = f(x_i, y_j)$ of size $N \times N$ is shaped into a vector $N^2 \times 1$:

$$\begin{pmatrix} f_{11} \\ \vdots \\ f_{1N} \\ f_{21} \\ \vdots \\ f_{2N} \\ \vdots \\ \vdots \\ \vdots \\ f_{NN} \end{pmatrix} \quad (7)$$

3 The three spatial derivative matrices?

With the ordering and number of grid points above, you should get a system of $N^2 \times 1$ equations. With given boundary conditions, the derivative operator matrix elements are computed using the finite difference schemes above to give the $N^2 \times N^2$ operator matrices:

$$A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla^2 \quad (8)$$

$$B = \frac{\partial}{\partial x} \quad (9)$$

$$C = \frac{\partial}{\partial y} \quad (10)$$

NOTE: You have to include the Δx and Δy terms that would appear in the denominator for all three matrices

4 The structure of the spatial derivative matrices?

Typical of finite-difference schemes, these matrices are sparse. The structure of these matrices for $N = 4$ is presented in figure 1, with the number of zeros indicated below the sparse structures. This figure was created using the MATLAB `'spy'` command.

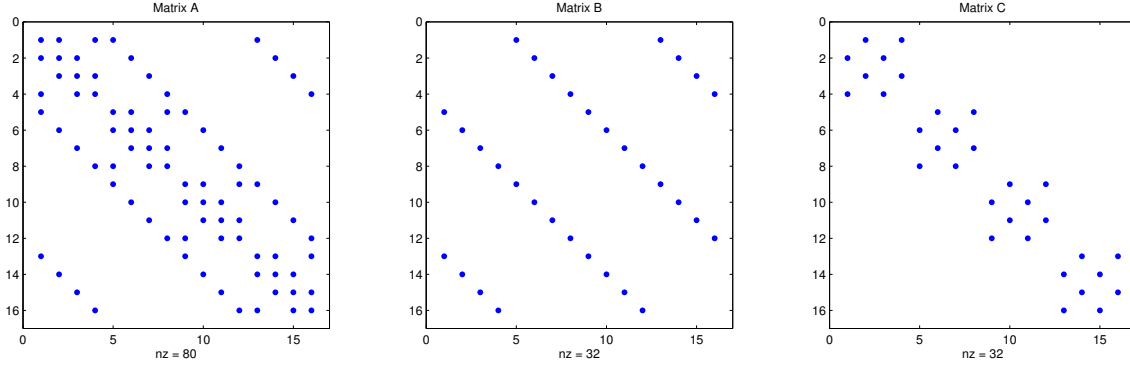


Figure 1: Sparse structure of first and second order spatial derivative matrices derived from Finite-difference methods

NOTE: Conventionally the main diagonal is indexed 0, the diagonal that lies above the main diagonal (upper first diagonal) is indexed as 1, the upper second diagonal index as 2 and so on. Similarly the lower diagonals are indexed -1, -2 and so on.

5 The discretized coupled equations?

Equations equation (1) and equation (2) after the above discretization take the form:

$$\omega_t = \nu A\psi - B\psi C\omega + C\psi B\omega \quad (11)$$

$$A\psi = \omega \quad (12)$$

Again, this is a system of $N^2 \times 1$ equations.