

Practice Problems 7: Solutions

1. Consider using Lagrange's Method to solve the following optimization problem:

$$\begin{aligned} \text{minimize:} \quad & 3x_1 - 4x_2 + x_3 - 2x_4 \\ \text{subject to:} \quad & -x_2^2 + x_3^2 + x_4^2 = 1 \\ & 3x_1^2 + x_3^2 + 2x_4^2 = 6 \end{aligned}$$

$$F(x, \lambda) = 3x_1 - 4x_2 + x_3 - 2x_4 + \lambda_1 (-x_2^2 + x_3^2 + x_4^2 - 1) + \lambda_2 (3x_1^2 + x_3^2 + 2x_4^2 - 6)$$

$$DF(x, \lambda) = \begin{bmatrix} 3 + 6\lambda_2 x_1 \\ -4 - 2\lambda_1 x_2 \\ 1 + 2\lambda_1 x_3 + 2\lambda_2 x_3 \\ -2 + 2\lambda_1 x_4 + 4\lambda_2 x_4 \\ -x_2^2 + x_3^2 + x_4^2 - 1 \\ 3x_1^2 + x_3^2 + 2x_4^2 - 6 \end{bmatrix}^T$$

Finding a critical point, solving $DF(x, \lambda) = 0$, is difficult because $DF(x, \lambda)$ is not a linear system.

2. Compute the gradient of the Lagrangian for the maximum expected return portfolio subject to risk = $(20\%)^2$

$$\begin{aligned} \text{maximize:} \quad & \mu^T w \\ \text{subject to:} \quad & e^T w = 1 \\ & w^T \Sigma w = (0.20)^2 \end{aligned}$$

$$F(w, \lambda) = \mu^T w + \lambda_1 (e^T w - 1) + \lambda_2 (w^T \Sigma w - 0.04)$$

$$DF(w, \lambda) = \begin{bmatrix} \mu^T + \lambda_1 e^T + 2\lambda_2 w^T \Sigma \\ e^T w - 1 \\ w^T \Sigma w - 0.04 \end{bmatrix}^T$$

Note: Unlike the lecture example of finding the minimum variance portfolio, $DF(w, \lambda) = 0$ is not a linear system in this case, so the solution must be computed numerically.

3. Second order Taylor polynomial around S_0 for $C(S)$ in terms of Δ and Γ :

$$C(S_0) + (S - S_0) \frac{\partial C}{\partial S}(S_0) + \frac{(S - S_0)^2}{2} \frac{\partial^2 C}{\partial S^2}(S_0) = C(S_0) + (S - S_0) \Delta(S_0) + \frac{(S - S_0)^2}{2} \Gamma(S_0)$$

$$4. \quad f(x) = \frac{1}{1+x} \\ f'(x) = \frac{-1}{(1+x)^2} \quad f''(x) = \frac{2}{(1+x)^3} \quad f'''(x) = \frac{-6}{(1+x)^4} \quad \dots \quad f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

Taylor series expansion:

$$T(x) = \frac{1}{1+0} + (x-0) \frac{-1}{(1+0)^2} + \frac{(x-0)^2}{2} \frac{2}{(1+0)^3} + \frac{(x-0)^3}{6} \frac{-6}{(1+0)^4} + \dots + \frac{(x-0)^n}{n!} \frac{(-1)^n n!}{(1+x)^{n+1}} \\ = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n$$

Radius of convergence:

$$R = \frac{1}{\lim_{k \rightarrow \infty} |a_k|^{1/k}} = \frac{1}{1} = 1 \quad \text{where} \quad \lim_{k \rightarrow \infty} |a_k|^{1/k} = \lim_{k \rightarrow \infty} |(-1)^k|^{1/k} = \lim_{k \rightarrow \infty} 1^{1/k} = 1$$

Where does $T(x) = \frac{1}{1+x}$?

$$\lim_{n \rightarrow \infty} \frac{r^n}{n!} \max_{z \in [-r, r]} |f^{(n)}(z)| = \lim_{n \rightarrow \infty} \frac{r^n}{n!} \max_{z \in [-r, r]} \left| \frac{(-1)^n n!}{(1+z)^{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{r^n}{(1-r)^{n+1}} = 0 \text{ for } r < \frac{1}{2}$$

$$T(x) = \frac{1}{1+x} \text{ for } |x| < \frac{1}{2}$$

$$5. \quad R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \Rightarrow R_t P_{t-1} = P_t - P_{t-1} \Rightarrow R_t P_{t-1} + P_{t-1} = P_t \Rightarrow P_t = (R_t + 1) P_{t-1}$$

$$r_t = \log \left(\frac{P_t}{P_{t-1}} \right) = \log \left(\frac{(R_t + 1) P_{t-1}}{P_{t-1}} \right) = \log(R_t + 1)$$

Note: R_t is actually a first order Taylor polynomial of r_t around the point $a = 0$.

Let $R_t = 0.01$, then $r_t = \log(1.01) \approx 0.00995$

6. Find the order of the Taylor polynomial needed to compute $e^{0.25}$ to six digits of accuracy.

$$f(x) - P_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt$$

Let $a = 0$ and $x = 0.25$

$$\text{Find } n \text{ such that } f(0.25) - P_n(0.25) = \int_0^{0.25} \frac{(0.25-t)^n}{n!} e^t dt < 10^{-6}$$

n	Error
1	3.40×10^{-2}
2	2.78×10^{-3}
3	1.73×10^{-4}
4	8.49×10^{-6}
5	3.52×10^{-7}

We need a 5th order Taylor polynomial to compute $e^{0.25}$ to six digits of accuracy.