

Practice Problems 6: Solutions

1. (a) Let A be an $m \times n$ matrix ($m \geq n$) with linearly independent columns:

Full QR factorization: $A = QR$

- Q is an $m \times m$ orthogonal matrix
- R is an $m \times n$ upper triangular matrix

Reduced QR factorization: $A = \tilde{Q}\tilde{R}$

- $\tilde{Q}$ is an $m \times n$ matrix with $\tilde{Q}^T \tilde{Q} = I$
- $\tilde{R}$ is an $n \times n$ upper triangular matrix

$$\begin{aligned}
 \text{(b) } H &= A(A^T A)^{-1} A^T = \tilde{Q}\tilde{R} \left((\tilde{Q}\tilde{R})^T (\tilde{Q}\tilde{R}) \right)^{-1} (\tilde{Q}\tilde{R})^T \\
 &= \tilde{Q}\tilde{R} \left(\tilde{R}^T \tilde{Q}^T \tilde{Q} \tilde{R} \right)^{-1} \tilde{R}^T \tilde{Q}^T \\
 &= \tilde{Q}\tilde{R} \left(\tilde{R}^T \tilde{R} \right)^{-1} \tilde{R}^T \tilde{Q}^T \\
 &= \tilde{Q}\tilde{R}\tilde{R}^{-1} \tilde{R}^{-T} \tilde{R}^T \tilde{Q}^T \\
 &= \tilde{Q}\tilde{Q}^T \\
 &= I
 \end{aligned}$$

Only two matrices are inverted: $\tilde{R}$ and $\tilde{R}^T$.

2. Consider the matrix $Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.

An eigenvector x must be in the same direction as Qx , that is, $Qx = \lambda x$. However, multiplying Q by a real vector x rotates the vector in the xy -plane through the angle θ , so x cannot be in the same direction as Qx .

3. (a) Let X be an $m \times n$ matrix:

Full singular value factorization $X = U\Sigma V^T$

- U is an $m \times m$ orthogonal matrix
- Σ is an $m \times n$ diagonal matrix
- V is an $n \times n$ orthogonal matrix

$$\begin{aligned}
 \tilde{\beta} &= (X^T X)^{-1} X^T y = \left((U\Sigma V^T)^T U\Sigma V^T \right)^{-1} (U\Sigma V^T)^T y \\
 &= (V\Sigma^T U^T U\Sigma V^T)^{-1} V\Sigma^T U^T y \\
 &= (V\Sigma^T \Sigma V^T)^{-1} V\Sigma^T U^T y \\
 &= V(\Sigma^T \Sigma)^{-1} V^T V\Sigma U^T y \\
 &= V(\Sigma^T \Sigma)^{-1} \Sigma U^T y
 \end{aligned}$$

Note: While it may be tempting to say that because Σ is a diagonal matrix $\Sigma = \Sigma^T$, keep in mind that Σ is an $m \times n$ matrix. If $m \neq n$, then Σ is not square, so it can't be symmetric. Furthermore, when Σ isn't square, it cannot be invertible, so $(\Sigma^T \Sigma)^{-1} \neq \Sigma^{-1} (\Sigma^T)^{-1}$.

(b) Reduced singular value factorization $X = \tilde{U} \tilde{\Sigma} \tilde{V}^T$

- $\tilde{U}$ is an $m \times n$ matrix with $\tilde{U}^T \tilde{U} = I$
- $\tilde{\Sigma}$ is an $n \times n$ diagonal matrix
- $\tilde{V}$ is an $n \times n$ orthogonal matrix

$$\begin{aligned} \tilde{\beta} &= (X^T X)^{-1} X^T y = \left((\tilde{U} \tilde{\Sigma} \tilde{V}^T)^T \tilde{U} \tilde{\Sigma} \tilde{V}^T \right)^{-1} (\tilde{U} \tilde{\Sigma} \tilde{V}^T)^T y \\ &= (\tilde{V} \tilde{\Sigma} \tilde{U}^T \tilde{U} \tilde{\Sigma} \tilde{V}^T)^{-1} \tilde{V} \tilde{\Sigma} \tilde{U}^T y \\ &= (\tilde{V} \tilde{\Sigma}^2 \tilde{V}^T)^{-1} \tilde{V} \tilde{\Sigma} \tilde{U}^T y \\ &= \tilde{V} \tilde{\Sigma}^{-2} \tilde{V}^T \tilde{V} \tilde{\Sigma} \tilde{U}^T y \\ &= \tilde{V} \tilde{\Sigma}^{-1} \tilde{U}^T y \end{aligned}$$

Note: $\tilde{\Sigma}^{-1}$ is actually $\tilde{\Sigma}^\dagger$, the pseudoinverse of $\tilde{\Sigma}$. This matrix is created by replacing all the non-zero diagonal entries of $\tilde{\Sigma}$ and then transposing the resulting matrix. If $\tilde{\Sigma}$ has full rank, then it is invertible and $\tilde{\Sigma}^\dagger = \tilde{\Sigma}^{-1}$. In addition, the expression for $\tilde{\beta}$ contains the pseudoinverse of matrix X : $X^\dagger = (X^T X)^{-1} X^T = \tilde{V} \tilde{\Sigma}^\dagger \tilde{U}^T$.

$$4. \quad \hat{\Sigma} = \frac{1}{m-1} \tilde{X}^T \tilde{X} = \frac{1}{m-1} (\tilde{Q} \tilde{R})^T (\tilde{Q} \tilde{R}) = \frac{1}{m-1} \tilde{R}^T \tilde{Q}^T \tilde{Q} \tilde{R} = \frac{1}{m-1} \tilde{R}^T \tilde{R}$$

$$d_i^2 = \tilde{x}_u^T \hat{\Sigma}^{-1} \tilde{x}_i = \tilde{x}_i^T \left(\frac{1}{m-1} \tilde{R}^T \tilde{R} \right)^{-1} \tilde{x}_i$$