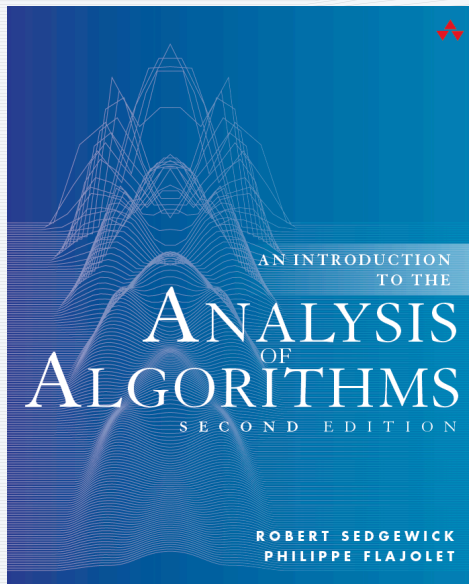


ANALYTIC COMBINATORICS

PART ONE



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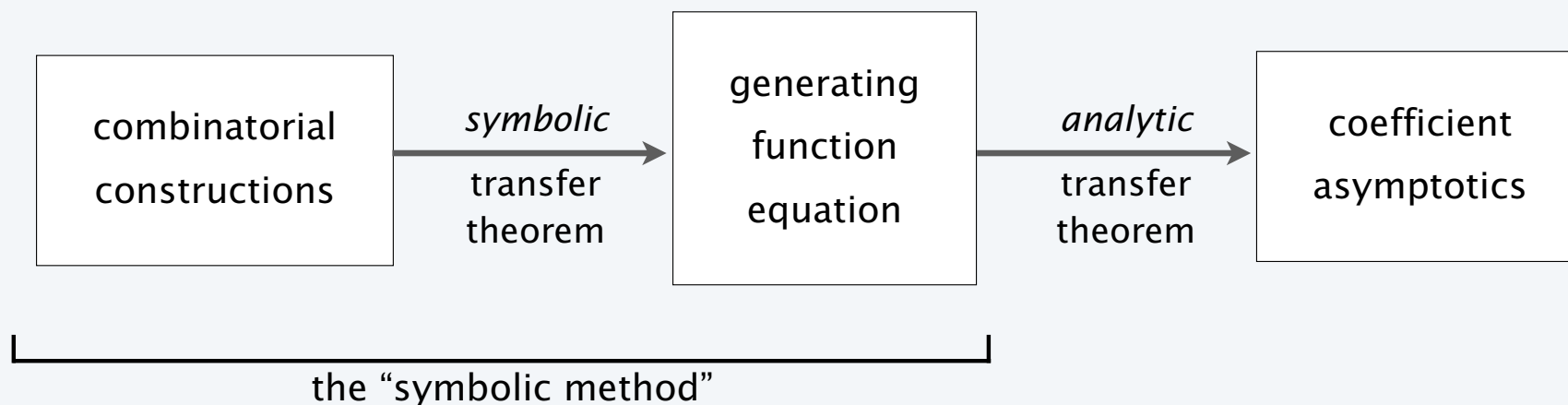
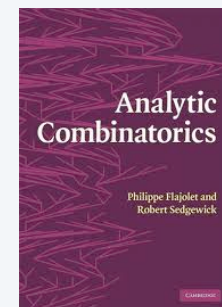
5. Analytic Combinatorics

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

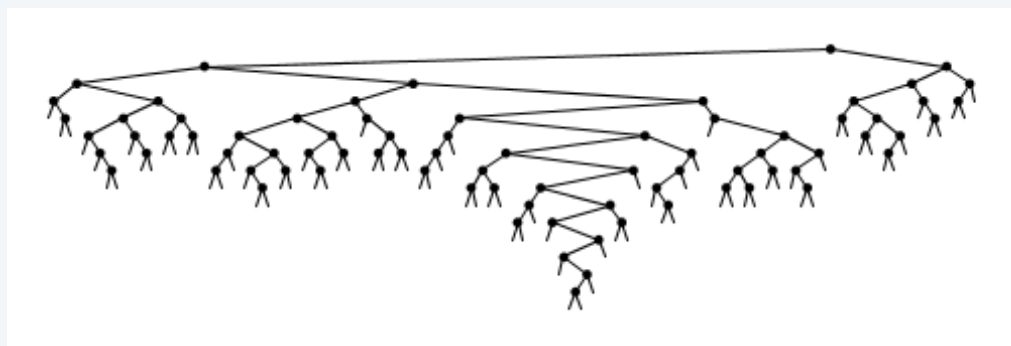
- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.



Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = 1 + zT(z)^2$$

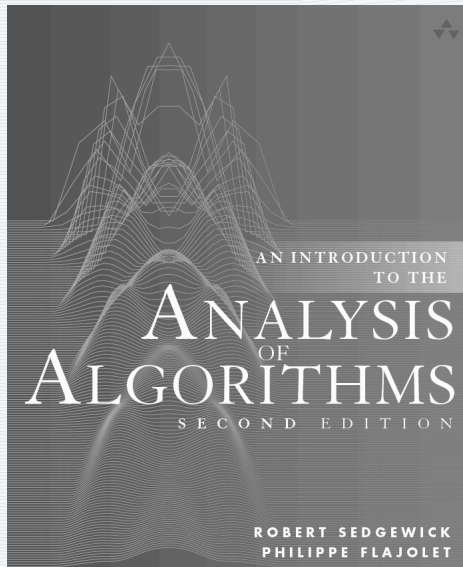
GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient
asymptotics

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5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

5a.AC.Symbolic

The symbolic method

is an approach for translating *combinatorial constructions* to *GF equations*

- Define a *class* of combinatorial objects.
- Define a notion of *size*.
- Define a *GF* whose coefficients count objects of the same size.
- Define *operations* suitable for constructive definitions of objects.
- Develop *translations* from constructions to operations on GFs.

Examples

A, B, Z

$|b|$

$A(z)$

$A \times B$

$A(z)B(z)$

Formal basis:

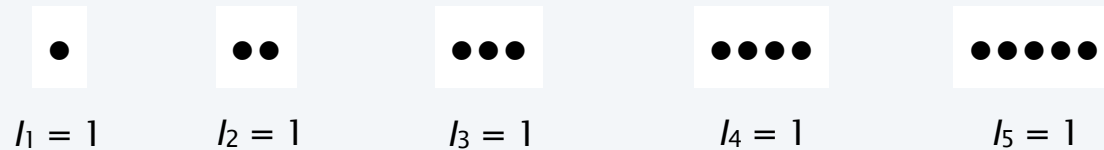
- A *combinatorial class* is a set of objects and a *size function*.
- An *atom* is an object of size 1.
- An *neutral object* is an atom of size 0.
- A *combinatorial construction* uses the union, product, and sequence operations to define a class in terms of atoms and other classes.

Building blocks

<i>notation</i>	<i>denotes</i>	<i>contains</i>
Z	atomic class	an atom
E	neutral class	neutral object
Φ	empty class	nothing

Unlabelled class example 1: natural numbers

Def. A *natural number* is a **set** (or a sequence) of atoms.



unary notation

counting sequence

OGF

$l_N = 1$	$\frac{1}{1-z}$
-----------	-----------------

$$\sum_{N \geq 0} z^N = \frac{1}{1-z}$$

Unlabelled class example 2: bitstrings

Def. A *bitstring* is a **sequence** of 0 or 1 bits.

$B_0 = 1$

$B_1 = 2$

$B_2 = 4$

$B_3 = 8$

$B_4 = 16$

counting sequence

OGF

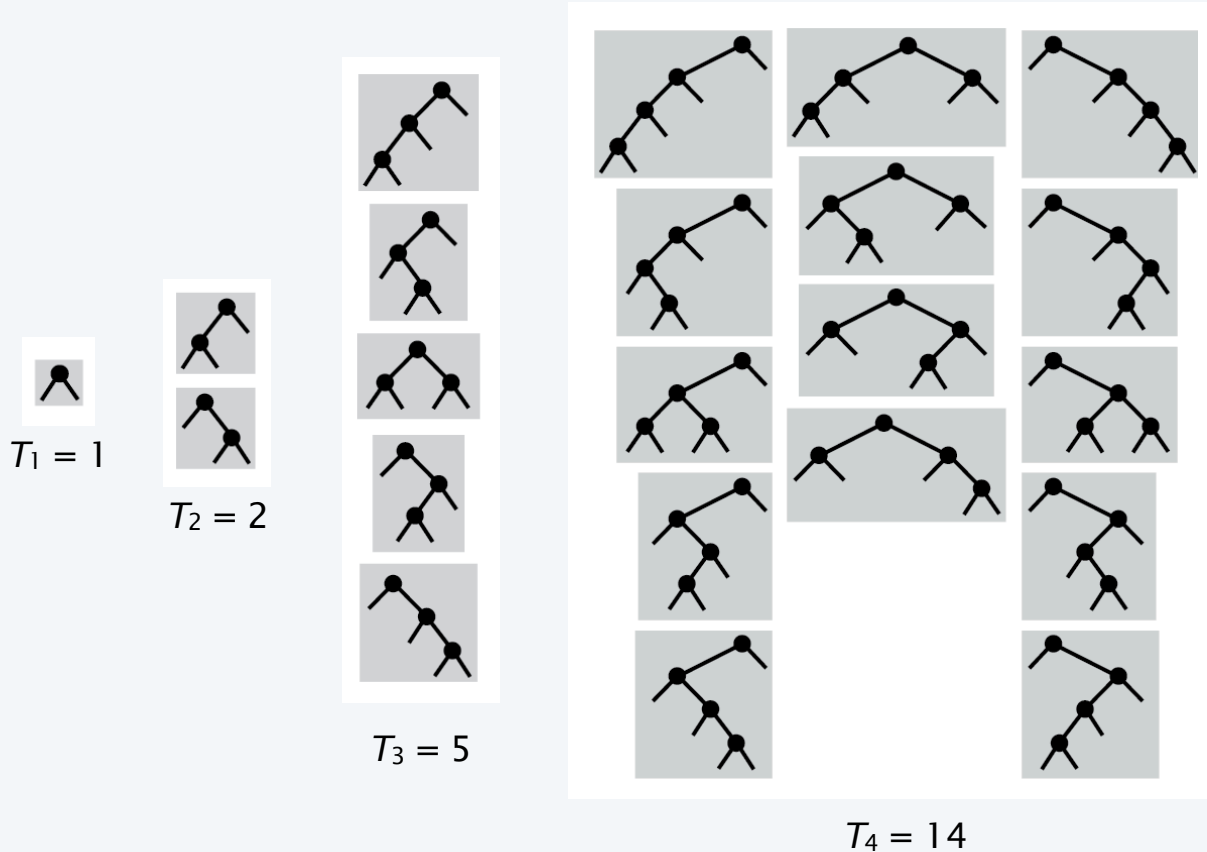
$$B_N = 2^N$$

$$\frac{1}{1-2z}$$

$$\sum_{N \geq 0} 2^N z^N = \sum_{N \geq 0} (2z)^N = \frac{1}{1-2z}$$

Unlabelled class example 3: binary trees

Def. A *binary tree* is empty or a **sequence** of a node and two binary trees



counting sequence

OGF

$T_N = \frac{1}{N+1} \binom{2N}{N}$	$\frac{1}{2z}(1 - \sqrt{1 - 4z})$
-------------------------------------	-----------------------------------

Catalan numbers (see Lecture 3)

$$T(z) = 1 + zT(z)^2$$

Combinatorial constructions for unlabelled classes

<i>construction</i>	<i>notation</i>	<i>semantics</i>
disjoint union	$A + B$	disjoint copies of objects from A and B
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B
sequence	$SEQ(A)$	sequences of objects from A

A and B are
combinatorial classes
of unlabelled objects

Ex 1. $(00 + 01) \times (101 + 110 + 111) = 00101 \ 00110 \ 00111 \ 01101 \ 01111$

Ex 2. $\bullet \times SEQ(\bullet) = \bullet \ \bullet\bullet \ \bullet\bullet\bullet \ \bullet\bullet\bullet\bullet \ \bullet\bullet\bullet\bullet\bullet \ \bullet\bullet\bullet\bullet\bullet\bullet \ \bullet\bullet\bullet\bullet\bullet\bullet\bullet \ \dots$

Ex 3. $\square \times \bullet \times \begin{array}{c} \bullet \\ \swarrow \searrow \\ \square \quad \square \end{array} = \begin{array}{c} \bullet \\ \swarrow \searrow \\ \square \quad \bullet \\ \quad \swarrow \searrow \\ \quad \square \quad \square \end{array}$

"unlabelled" ?? Stay tuned.

The symbolic method for unlabelled classes (transfer theorem)

Theorem. Let A and B be combinatorial classes of unlabelled objects with OGFs $A(z)$ and $B(z)$. Then

<i>construction</i>	<i>notation</i>	<i>semantics</i>	<i>OGF</i>
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
Cartesian product	$A \times B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ(A)$	sequences of objects from A	$\frac{1}{1 - A(z)}$

Proofs of transfers

are immediate from GF counting

$A + B$

$$\sum_{\gamma \in A+B} z^{|\gamma|} = \sum_{\alpha \in A} z^{|\alpha|} + \sum_{\beta \in B} z^{|\beta|} = A(z) + B(z)$$

$A \times B$

$$\sum_{\gamma \in A \times B} z^{|\gamma|} = \sum_{\alpha \in A} \sum_{\beta \in B} z^{|\alpha|+|\beta|} = \left(\sum_{\alpha \in A} z^{|\alpha|} \right) \left(\sum_{\beta \in B} z^{|\beta|} \right) = A(z)B(z)$$

$SEQ(A)$

$$SEQ(A) \equiv \epsilon + A + A^2 + A^3 + A^4 + \dots$$

$$1 + A(z) + A(z)^2 + A(z)^3 + A(z)^4 + \dots = \frac{1}{1 - A(z)}$$

Symbolic method: binary trees

How many **binary trees** with N nodes?

<i>Class</i>	T , the class of all binary trees
<i>Size</i>	$ t $, the number of internal nodes in t
<i>OGF</i>	$T(z) = \sum_{t \in T} z^{ t } = \sum_{N \geq 0} T_N z^N$

Atoms

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
external node	$Z_{\square}$	0	1
internal node	$Z_{\bullet}$	1	z

Construction

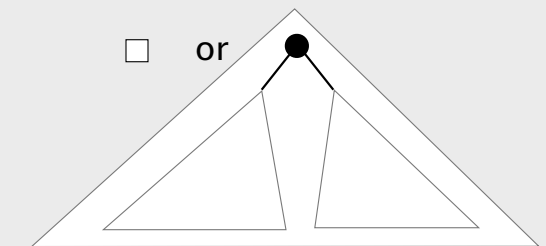
$$T = Z_{\square} + T \times Z_{\bullet} \times T$$

OGF equation

$$T(z) = 1 + zT(z)^2$$

$$[z^N]T(z) = \frac{1}{N+1} \binom{2N}{N} \sim \frac{4^N}{\sqrt{\pi N^3}}$$

“a binary tree is an external node or an internal node connected to two binary trees”



see Lecture 3 and stay tuned.

Symbolic method: binary trees

How many binary trees with N *external* nodes?

Class	T , the class of all binary trees
Size	$\boxed{t}$, the number of <i>external</i> nodes in t
OGF	$T^{\square}(z) = \sum_{t \in T} z^{\boxed{t}}$

Atoms

type	class	size	GF
external node	$Z_{\square}$	1	z
internal node	$Z_{\bullet}$	0	1

Construction

$$T = Z_{\square} + T \times Z_{\bullet} \times T$$

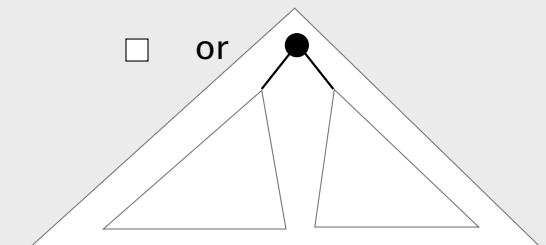
OGF equation

$$T^{\square}(z) = z + T^{\square}(z)^2$$

$$T^{\square}(z) = zT(z)$$

$$[z^N]T^{\square}(z) = [z^{N-1}]T(z) = \frac{1}{N} \binom{2N-2}{N-1}$$

“a binary tree is an external node or an internal node connected to two binary trees”



← same as # binary trees with $N-1$ internal nodes

Symbolic method: binary strings

Warmup: How many **binary strings** with N bits?

<i>Class</i>	B , the class of all binary strings
<i>Size</i>	$ b $, the number of bits in b
<i>OGF</i>	$B(z) = \sum_{b \in B} z^{ b } = \sum_{N \geq 0} B_N z^N$

Atoms

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
0 bit	Z_0	1	z
1 bit	Z_1	1	z

Construction

$$B = \text{SEQ}(Z_0 + Z_1)$$

“a binary string is a sequence of 0 bits and 1 bits”

OGF equation

$$B(z) = \frac{1}{1 - 2z}$$

$$[z^N]B(z) = 2^N \quad \checkmark$$

Symbolic method: binary strings (alternate)

Warmup: How many **binary strings** with N bits?

<i>Class</i>	B , the class of all binary strings
<i>Size</i>	$ b $, the number of bits in b
<i>OGF</i>	$B(z) = \sum_{b \in B} z^{ b } = \sum_{N \geq 0} B_N z^N$

Atoms

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
0 bit	Z_0	1	z
1 bit	Z_1	1	z

Construction

$$B = E + (Z_0 + Z_1) \times B$$

“a binary string is empty or a bit followed by a binary string”

OGF equation

$$B(z) = 1 + 2zB(z)$$

Solution

$$B(z) = \frac{1}{1 - 2z}$$

$$[z^N]B(z) = 2^N \quad \checkmark$$

Symbolic method: binary strings with restrictions

Ex. How many N -bit binary strings have **no two consecutive 0s**?

□

$T_0 = 1$

0
1

$T_1 = 2$

0 1
1 0
1 1

$T_2 = 3$

0 1 1
0 1 0
1 0 1
1 1 0
1 1 1

$T_3 = 5$

0 1 0 1
0 1 1 0
0 1 1 1
1 0 1 1
1 0 1 0
1 1 0 1
1 1 1 0
1 1 1 1

$T_4 = 8$

0 1 0 1 1
0 1 0 1 0
0 1 1 0 1
0 1 1 1 0
0 1 1 1 1
1 0 1 0 1
1 0 1 1 0
1 0 1 1 1
1 1 0 1 1
1 1 0 1 0
1 1 1 0 1
1 1 1 1 0
1 1 1 1 1

$T_5 = 13$

Stay tuned for general treatment (Chapter 8)

Symbolic method: binary strings with restrictions

Ex. How many N -bit binary strings have **no two consecutive 0s**?

<i>Class</i>	B_{00} , the class of binary strings with no 00
<i>Size</i>	$ b $, the number of bits in b
<i>OGF</i>	$B_{00}(z) = \sum_{b \in B_{00}} z^{ b }$

Atoms

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
0 bit	Z_0	1	z
1 bit	Z_1	1	z

Construction $B_{00} = E + Z_0 + (Z_1 + Z_0 \times Z_1) \times B_{00}$

“a binary string with no 00 is either empty or 0 or it is 1 or 01 followed by a binary string with no 00”

OGF equation $B_{00}(z) = 1 + z + (z + z^2)B_{00}(z)$

solution $B_{00}(z) = \frac{1+z}{1-z-z^2}$

$$[z^N]B_{00}(z) = F_N + F_{N+1} = F_{N+2}$$

1, 2, 5, 8, 13, ... ✓

Symbolic method: many, many examples to follow

How many ... with ... ?

<i>Class</i>	
<i>Size</i>	
<i>OGF</i>	

Atoms

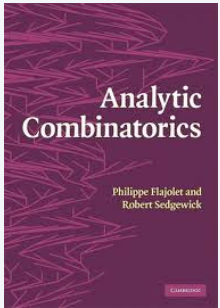
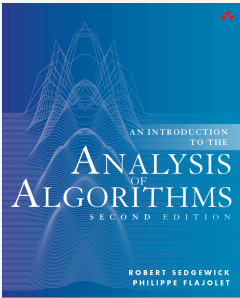
<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>

Construction

OGF equation

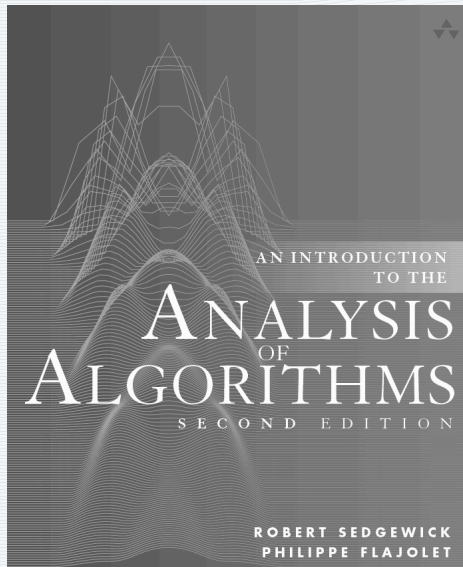
solution

“a ... is either ...
or ... and ...”



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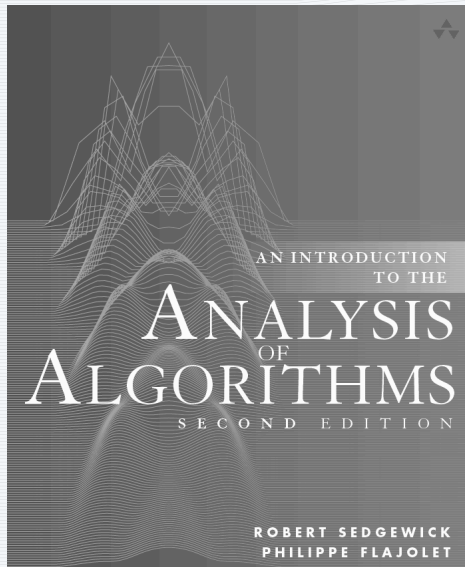
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- Perspective

5a.AC.Symbolic

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5. Analytic Combinatorics

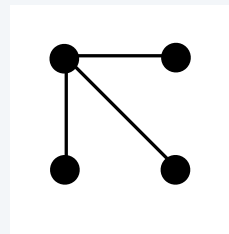
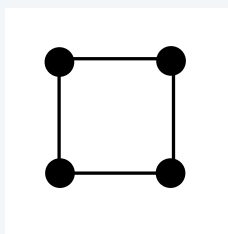
- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
- Perspective

5b.AC.Labelled

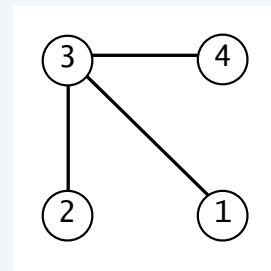
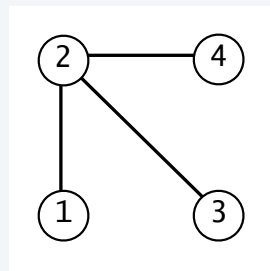
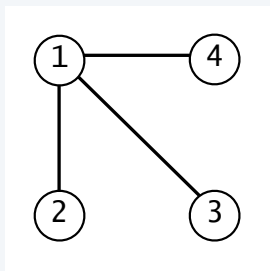
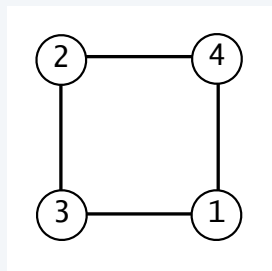
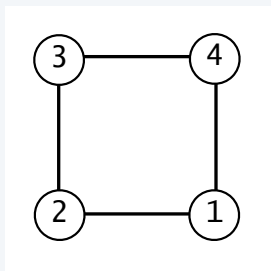
Labelled combinatorial classes

have objects composed of N atoms, labelled with the integers 1 through N .

Ex. Different unlabelled objects

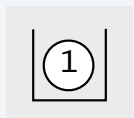


Ex. Different labelled objects

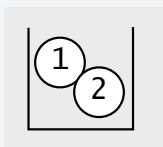


Labelled class example 1: urns

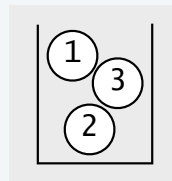
Def. An *urn* is a **set** of labelled atoms.



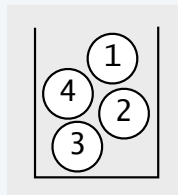
$$U_1 = 1$$



$$U_2 = 1$$



$$U_3 = 1$$



$$U_4 = 1$$

counting sequence

EGF

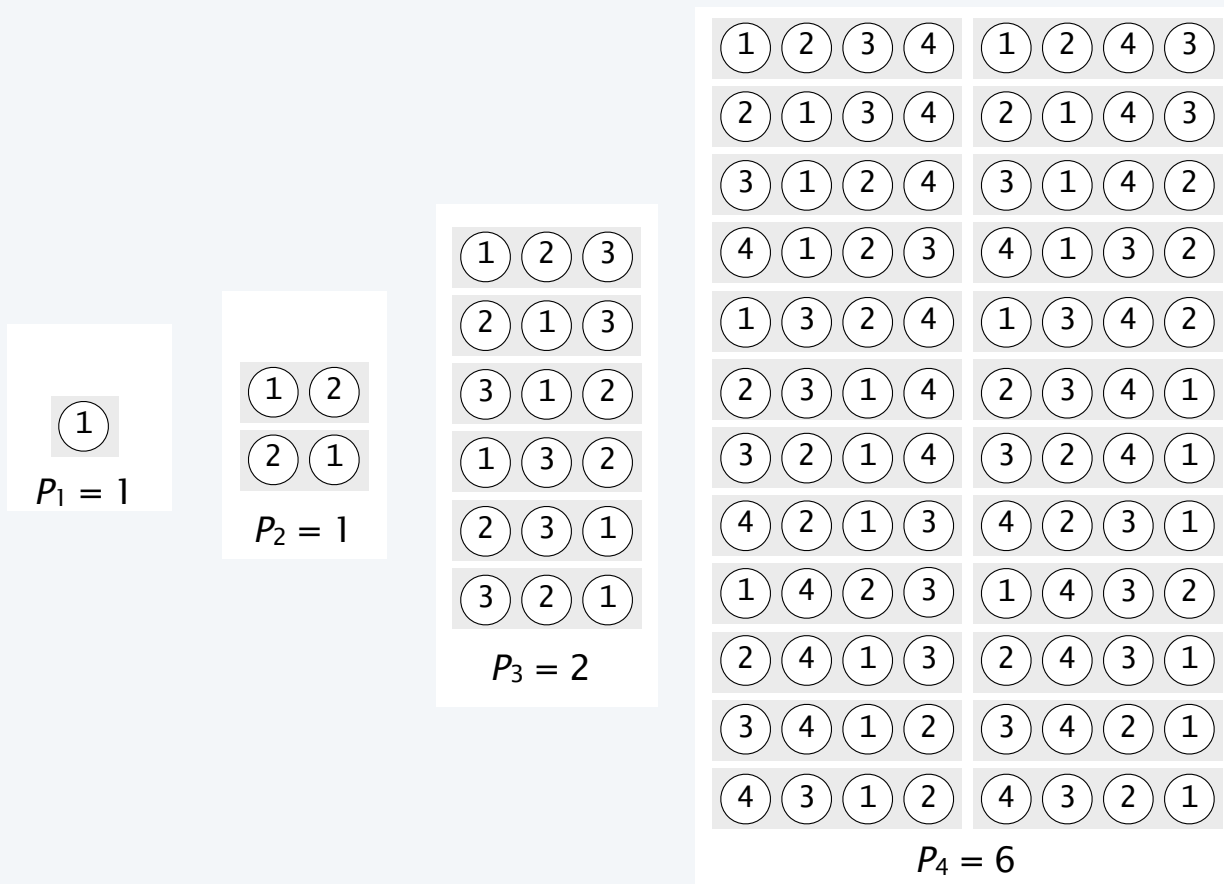
$$U_N = 1$$

$$e^z$$

$$\sum_{N \geq 0} \frac{z^N}{N!} = e^z$$

Labelled class example 2: permutations

Def. A *permutation* is a **sequence** of labelled atoms.



counting sequence

$$P_N = N!$$

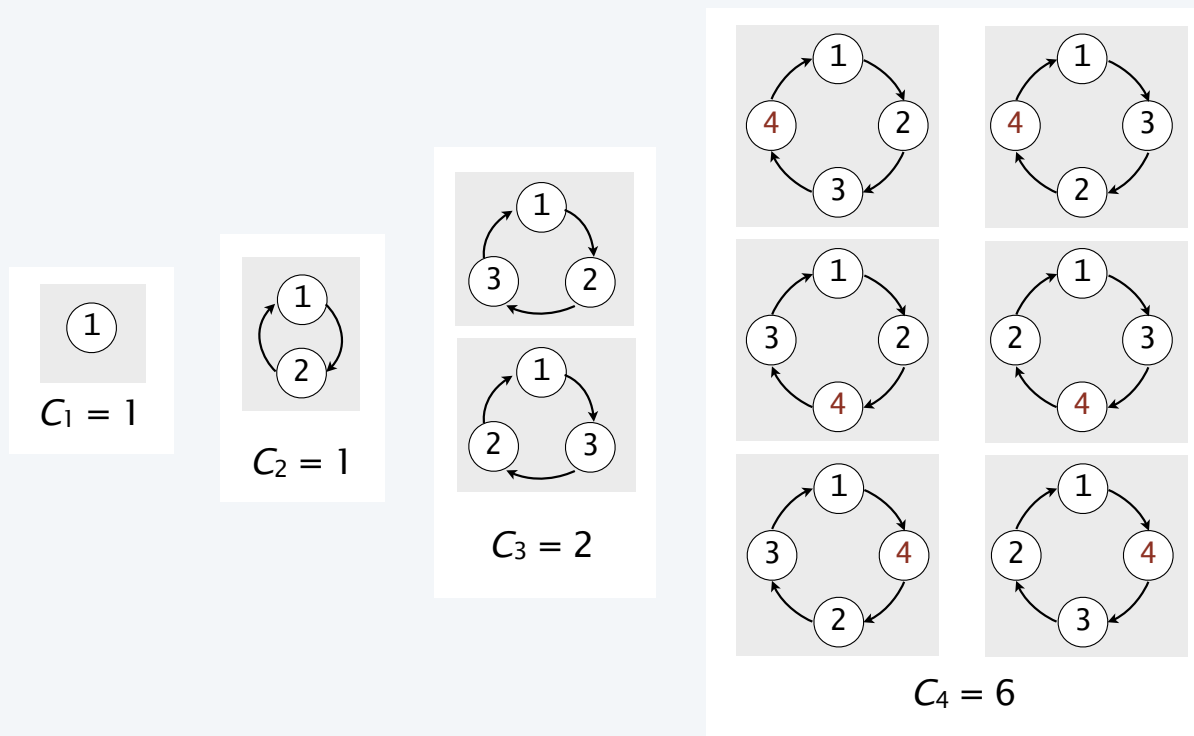
EGF

$$\frac{1}{1-z}$$

$$\sum_{N \geq 0} \frac{N! z^N}{N!} = \sum_{N \geq 0} z^N = \frac{1}{1-z}$$

Labelled class example 3: cycles

Def. A *cycle* is a **cyclic sequence** of labelled atoms



counting sequence

EGF

$$C_N = (N - 1)!$$

$$\ln \frac{1}{1 - z}$$

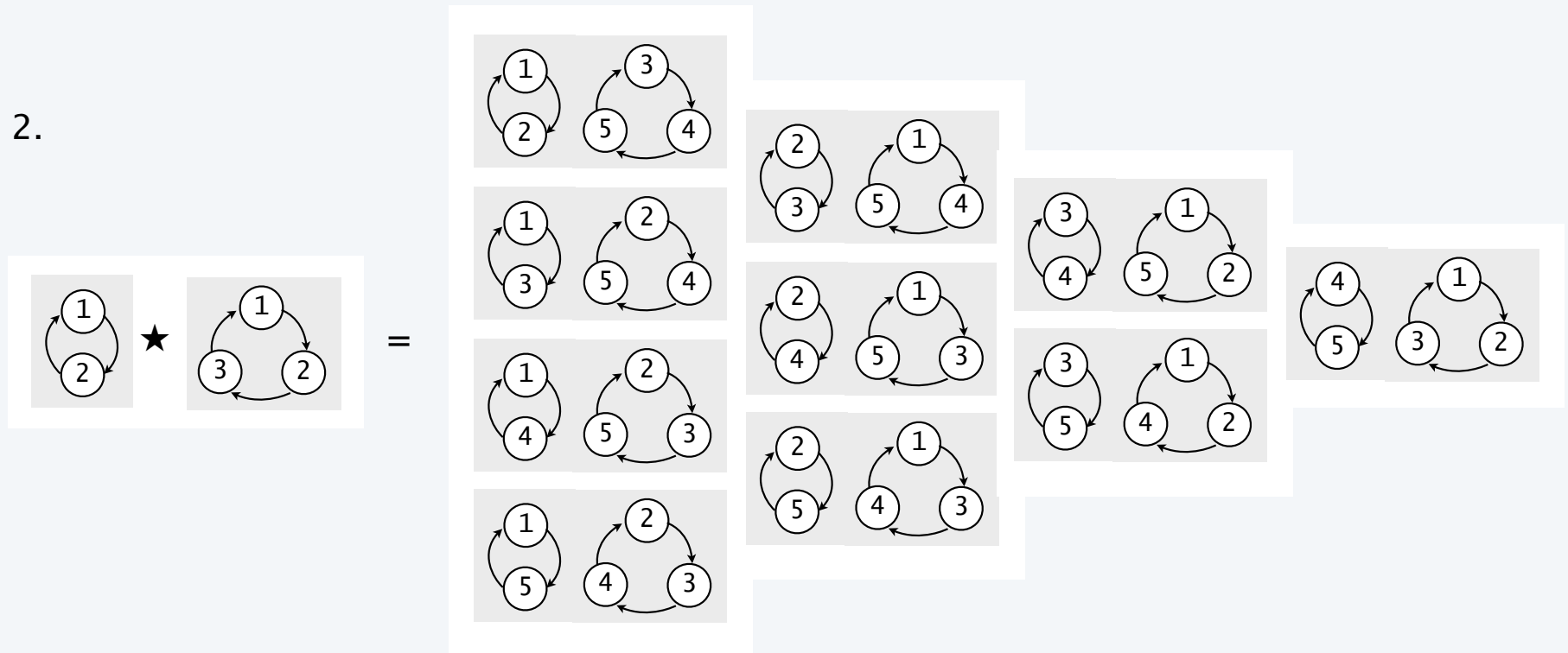
$$\sum_{N \geq 1} \frac{(N - 1)! z^N}{N!} = \sum_{N \geq 1} \frac{z^N}{N} = \ln \frac{1}{1 - z}$$

Star product operation

Analog to Cartesian product requires *relabelling in all consistent ways*.

Ex 1. $\boxed{1} \star \boxed{1 \ 2 \ 3} = \boxed{1 \ 2 \ 3 \ 4} \ \boxed{2 \ 1 \ 3 \ 4} \ \boxed{3 \ 1 \ 2 \ 4} \ \boxed{4 \ 1 \ 2 \ 3}$

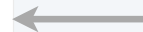
Ex 2.



Combinatorial constructions for labelled classes

<i>construction</i>	<i>notation</i>	<i>semantics</i>
disjoint union	$A + B$	disjoint copies of objects from A and B
labelled product	$A \star B$	ordered pairs of copies of objects, one from A and one from B
sequence	$SEQ(A)$	sequences of objects from A
set	$SET(A)$	sets of objects from A
cycle	$CYC(A)$	cyclic sequences of objects from A

A and B are
combinatorial classes
of labelled objects



The symbolic method for labelled classes (transfer theorem)

Theorem. Let A and B be combinatorial classes of **labelled** objects with **EGFs** $A(z)$ and $B(z)$. Then

<i>construction</i>	<i>notation</i>	<i>semantics</i>	<i>EGF</i>
disjoint union	$A + B$	disjoint copies of objects from A and B	$A(z) + B(z)$
labelled product	$A \star B$	ordered pairs of copies of objects, one from A and one from B	$A(z)B(z)$
sequence	$SEQ_k(A)$	k -sequences of objects from A	$A(z)^k$
	$SEQ(A)$	sequences of objects from A	$\frac{1}{1 - A(z)}$
set	$SET_k(A)$	k -sets of objects from A	$A(z)^k/k!$
	$SET(A)$	sets of objects from A	$e^{A(z)}$
cycle	$CYC_k(A)$	k -cycles of objects from A	$A(z)^k/k$
	$CYC(A)$	cycles of objects from A	$\ln \frac{1}{1 - A(z)}$

The symbolic method for labelled classes: basic constructions

<i>class</i>	<i>construction</i>	<i>EGF</i>	<i>counting sequence</i>
urns	$U = SET(Z)$	$U(z) = e^z$	$U_N = 1$
cycles	$C = CYC(Z)$	$C(z) = \ln \frac{1}{1-z}$	$C_N = (N-1)!$
permutations	$P = SEQ(Z)$	$P(z) = \frac{1}{1-z}$	$P_N = N!$
	$P = E + Z \star P$		

<i>construction</i>	<i>notation</i>	<i>EGF</i>
disjoint union	$A + B$	$A(z) + B(z)$
labelled product	$A \star B$	$A(z)B(z)$
sequence	$SEQ_k(A)$	$A(z)^k$
	$SEQ(A)$	$\frac{1}{1-A(z)}$
set	$SET_k(A)$	$A(z)^k/k!$
	$SET(A)$	$e^{A(z)}$
cycle	$CYC_k(A)$	$A(z)^k/k$
	$CYC(A)$	$\ln \frac{1}{1-A(z)}$

Proofs of transfers

are immediate from GF counting

$A + B$

$$\sum_{\gamma \in A+B} \frac{z^{|\gamma|}}{|\gamma|!} = \sum_{\alpha \in A} \frac{z^{|\alpha|}}{|\alpha|!} + \sum_{\beta \in B} \frac{z^{|\beta|}}{|\beta|!} = A(z) + B(z)$$

$A \star B$

$$\sum_{\gamma \in \mathcal{A} \times \mathcal{B}} \frac{z^{|\gamma|}}{|\gamma|!} = \sum_{\alpha \in \mathcal{A}} \sum_{\beta \in \mathcal{B}} \binom{|\alpha| + |\beta|}{|\alpha|} \frac{z^{|\alpha|+|\beta|}}{(|\alpha| + |\beta|)!} = \left(\sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} \right) \left(\sum_{\beta \in \mathcal{B}} \frac{z^{|\beta|}}{|\beta|!} \right) = A(z)B(z)$$

Notation. We write A^2 for $A \star A$, A^3 for $A \star A \star A$, etc.

Proofs of transfers

are immediate from GF counting

$$A(z)^k = \sum_{N \geq 0} \{\#k\text{-sequences of size } N\} \frac{z^N}{N!} = \sum_{N \geq 0} k \{\#k\text{-cycles of size } N\} \frac{z^N}{N!} = \sum_{N \geq 0} k! \{\#k\text{-sets of size } N\} \frac{z^N}{N!}$$

$$\frac{A(z)^k}{k} = \sum_{N \geq 0} \{\#k\text{-cycles of size } N\} \frac{z^N}{N!}$$

$$\frac{A(z)^k}{k!} = \sum_{N \geq 0} \{\#k\text{-sets of size } N\} \frac{z^N}{N!}$$

<i>class</i>	<i>construction</i>	<i>EGF</i>
k-sequence	$SEQ_k(A)$	$A(z)^k$
sequence	$SEQ_k(A) = SEQ_0(A) + SEQ_1(A) + SEQ_2(A) + \dots$	$1 + A(z) + A(z)^2 + A(z)^3 + \dots = \frac{1}{1 - A(z)}$
k-cycle	$CYC_k(A)$	$\frac{A(z)^k}{k}$
cycle	$CYC_k(A) = CYC_0(A) + CYC_1(A) + CYC_2(A) + \dots$	$1 + \frac{A(z)}{1} + \frac{A(z)^2}{2} + \frac{A(z)^3}{3} + \dots = \ln \frac{1}{1 - A(z)}$
k-set	$SET_k(A)$	$\frac{A(z)^k}{k!}$
set	$SET_k(A) = SET_0(A) + SET_1(A) + SET_2(A) + \dots$	$1 + \frac{A(z)}{1!} + \frac{A(z)^2}{2!} + \frac{A(z)^3}{3!} + \dots = e^{A(z)}$

Labelled class example 4: sets of cycles

Q. How many **sets of cycles** of labelled atoms?

①

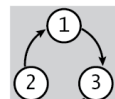
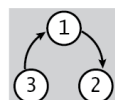
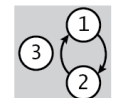
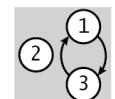
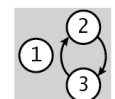
$$P_1^* = 1$$

① ②



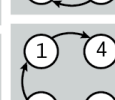
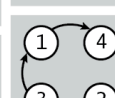
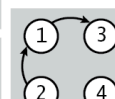
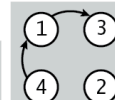
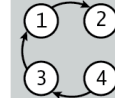
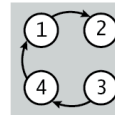
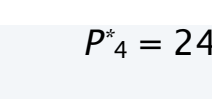
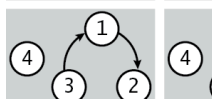
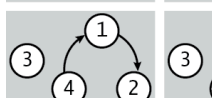
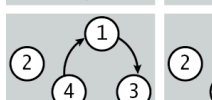
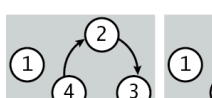
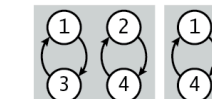
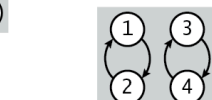
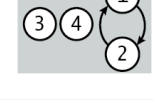
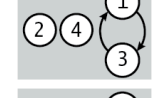
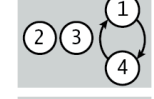
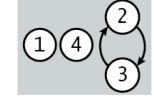
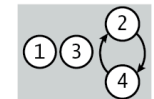
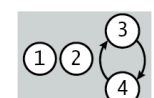
$$P_2^* = 2$$

① ② ③



$$P_3^* = 6$$

① ② ③ ④



$$P_4^* = 24$$

Symbolic method: sets of cycles

How many **sets of cycles** of length N ?

<i>Class</i>	P^* , the class of all sets of cycles of atoms
<i>Size</i>	$ p $, the number of atoms in p
<i>EGF</i>	$P^*(z) = \sum_{p \in P^*} \frac{z^{ p }}{ p !} = \sum_{N \geq 0} P_N^* \frac{z^N}{N!}$

Atom

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
labelled atom	Z	1	z

Construction

$$P^* = SET(CYC(Z))$$

OGF equation

$$P^*(z) = \exp\left(\ln \frac{1}{1-z}\right) = \frac{1}{1-z}$$

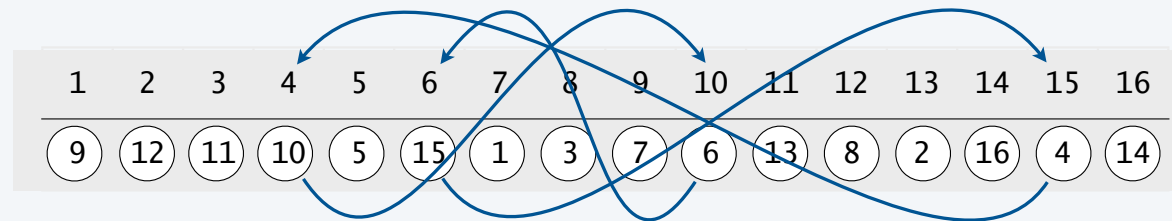
Counting sequence

$$P_N^* = N! [z^N] P^*(z) = N!$$

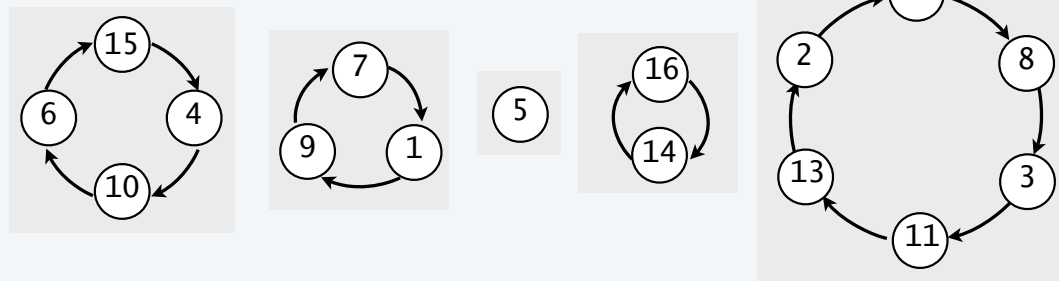
Aside: A combinatorial bijection

A permutation is a set of cycles.

Standard representation



Set of cycles representation



Derangements

N people go to the opera and leave their hats on a shelf in the cloakroom. When leaving, they each grab a hat at random.

Q. What is the probability that nobody gets their own hat?



Definition. A **derangement** is a permutation with no singleton cycles

Derangements (various versions)

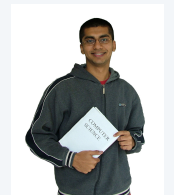
A group of N people go to the opera and leave their hats in the cloakroom. When leaving, they each grab a hat at random.

Q. What is the probability that nobody gets their own hat?



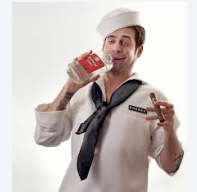
A professor returns exams to N students by passing them out at random.

Q. What is the probability that nobody gets their own exam?



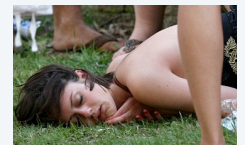
A group of N sailors go ashore for revelry that leads to a state of inebriation. When returning, they each end up sleeping in a random cabin.

Q. What is the probability that nobody sleeps in their own cabin?



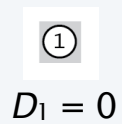
A group of N students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. What is the probability that nobody ends up in their own room?

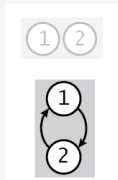


Derangements

are permutations with no singleton cycles.



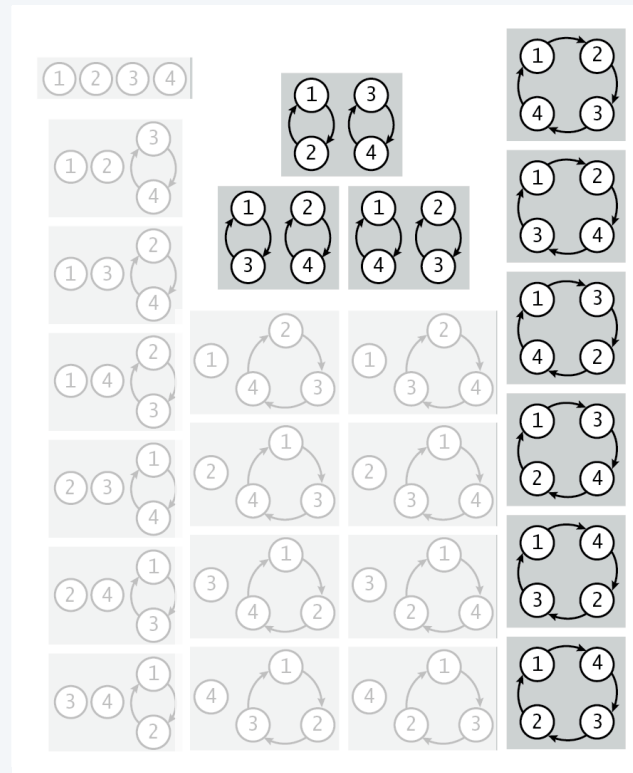
$$D_1 = 0$$



$$D_2 = 1$$



$$D_3 = 2$$



$$D_4 = 9$$

Symbolic method: derangements

How many **derangements** of length N ?

<i>Class</i>	D , the class of all derangements
<i>Size</i>	$ p $, the number of atoms in p
<i>EGF</i>	$D(z) = \sum_{d \in D} \frac{z^{ d }}{ d !} = \sum_{N \geq 0} D_N \frac{z^N}{N!}$

Atom

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
labelled atom	Z	1	z

Construction

$$D = SET(CYC_{>1}(Z))$$

"Derangements are permutations with no singleton cycles"

OGF equation

$$D(z) = e^{z^2/2 + z^3/3 + z^4/4 + \dots} = \exp\left(\ln \frac{1}{1-z} - z\right) = \frac{e^{-z}}{1-z}$$

Expansion

$$[z^N]D(z) \equiv \frac{D_N}{N!} = \sum_{0 \leq k \leq N} \frac{(-1)^k}{k!} \sim \left(\frac{1}{e}\right)$$

probability that a random permutation is a derangement

simple convolution

see "Asymptotics" lecture

Alternate derivation

$$\begin{aligned} Set(Z) \star D &= P \\ e^z D(z) &= \frac{1}{1-z} \end{aligned}$$

Derangements

A group of N students who live in single rooms go to a party that leads to a state of inebriation. When returning, they each end up in a random room.

Q. What is the probability that nobody ends up in their own room?



A. $\frac{1}{e} \doteq 0.36788$

Derangements

A group of N graduating seniors each throw their hats in the air and each catch a random hat.

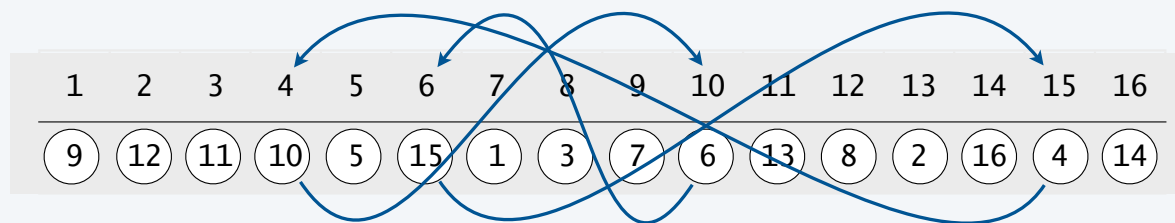
Q. What is the probability that nobody gets their own hat back?



A. $\frac{1}{e} \doteq 0.36788$

Generalized derangements

In the hats-in-the-air scenario, a student can get her hat back by "following the cycle".



Q. What is the probability that all cycles are of length $> M$?

Symbolic method: generalized derangements

How many permutations of length N have no cycles of length $\leq M$?

<i>Class</i>	D_M , the class of all generalized derangements
<i>Size</i>	$ d $, the number of atoms in d
<i>EGF</i>	$D_M(z) = \sum_{d \in D_M} \frac{z^{ d }}{ d !} = \sum_{N \geq 0} D_M N \frac{z^N}{N!}$

Atom

<i>type</i>	<i>class</i>	<i>size</i>	<i>GF</i>
labelled atom	Z	1	z

Construction

$$D_M = SET(CYC_{>M}(Z))$$

OGF equation

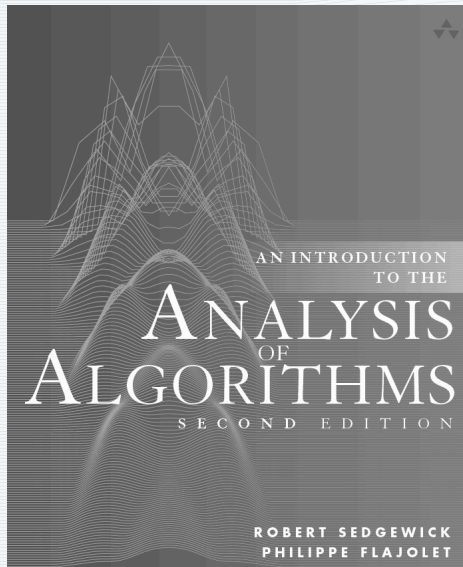
$$\begin{aligned}
 D_M(z) &= e^{\frac{z^{M+1}}{M+1} + \frac{z^{M+2}}{M+2} + \dots} = \exp\left(\ln \frac{1}{1-z} - z - z^2/2 - \dots - z^M/M\right) \\
 &= \frac{e^{-z - \frac{z^2}{2} - \frac{z^3}{3} - \dots - \frac{z^M}{M}}}{1-z}
 \end{aligned}$$

Expansion

$$D_{MN} = ?? \text{ } M\text{-way convolution (stay tuned)}$$

ANALYTIC COMBINATORICS

PART ONE



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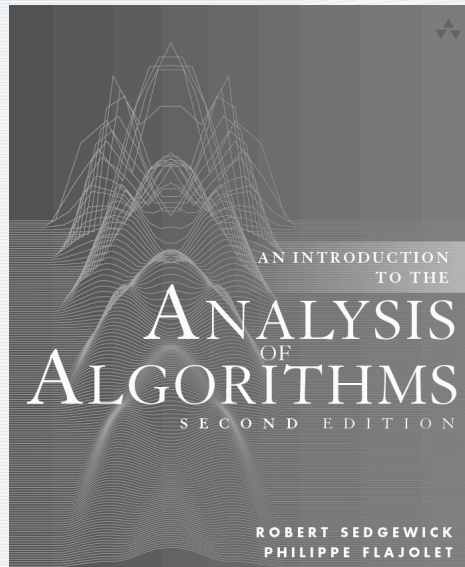
5. Analytic Combinatorics

- The symbolic method
- **Labelled objects**
- Coefficient asymptotics
- Perspective

5b.AC.Labelled

ANALYTIC COMBINATORICS

PART ONE



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5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- **Coefficient asymptotics**
- Perspective

5c.AC.Asymptotics

Generating coefficient asymptotics

are often *immediately* derived via general "analytic" transfer theorems.

Example 1. Taylor's theorem

Theorem. If $f(z)$ has N derivatives, then $[z^N]f(z) = f^{(N)}(0)/N!$

Example 2. Rational functions transfer theorem (see "Asymptotics" lecture)

Theorem. If $f(z)$ and $g(z)$ are polynomials, then

$$[z^n] \frac{f(z)}{g(z)} = -\frac{\beta f(1/\beta)}{g'(\beta)} \beta^n$$

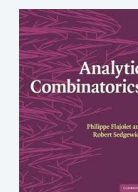
where $1/\beta$ is the largest root of g (provided that it has multiplicity 1).

see "Asymptotics"
lecture for general case

Example 3. Radius-of-convergence transfer theorem

[see next slide]

Most are based on
complex asymptotics.
Stay tuned for Part 2



Radius-of-convergence transfer theorem

Theorem. If $f(z)$ has radius of convergence >1 with $f(1) \neq 0$, then

$$[z^n] \frac{f(z)}{(1-z)^\alpha} \sim f(1) \binom{n+\alpha-1}{n} \sim \frac{f(1)}{\Gamma(\alpha)} n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \dots$

convolution,
 $f_1 + f_2 + \dots + f_n \sim f(1)$

standard asymptotics
with generalized
binomial coefficient

*Gamma function
(generalized factorial)*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$$

$$\Gamma(N + 1) = N!$$

$$\Gamma(1) = 1$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Corollary. If $f(z)$ has radius of convergence $>\rho$ with $f(\rho) \neq 0$, then

$$[z^n] \frac{f(z)}{(1-z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \dots$

Radius-of-convergence transfer theorem: applications

Corollary. If $f(z)$ has radius of convergence $>\rho$ with $f(\rho) \neq 0$, then

$$[z^n] \frac{f(z)}{(1 - z/\rho)^\alpha} \sim \frac{f(\rho)}{\Gamma(\alpha)} \rho^n n^{\alpha-1}$$

for any real $\alpha \notin 0, -1, -2, \dots$

Ex 1: Catalan

$$T(z) = \frac{1}{2z} (1 - \sqrt{1 - 4z})$$
$$[z^N] T(z) \sim \frac{4^N}{\sqrt{\pi N^3}}$$

$$\rho = 1/4 \quad \alpha = -1/2 \quad f(z) = -1/2$$
$$\Gamma(-1/2) = -2\Gamma(1/2) = -2\sqrt{\pi}$$

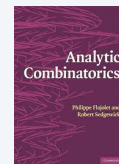
Ex 2: Derangements

$$D_M(z) = \frac{e^{-z-z^2/2 \dots - z^M/M}}{1 - z}$$
$$[z^N] D_M(z) \sim \frac{N!}{e^{H_M}}$$

$$\rho = 1 \quad \alpha = 1 \quad f(z) = e^{-z-z^2/2 \dots - z^M/M}$$

Transfer theorems based on complex asymptotics

provide *universal laws* of sweeping generality



Example: Context-free constructions

A system of combinatorial constructions

$$\begin{aligned} \langle \mathbf{G}_0 \rangle &= OP_0(\langle \mathbf{G}_0 \rangle, \langle \mathbf{G}_1 \rangle, \dots, \langle \mathbf{G}_t \rangle) \\ \langle \mathbf{G}_1 \rangle &= OP_1(\langle \mathbf{G}_0 \rangle, \langle \mathbf{G}_1 \rangle, \dots, \langle \mathbf{G}_t \rangle) \\ &\dots \\ \langle \mathbf{G}_t \rangle &= OP_t(\langle \mathbf{G}_0 \rangle, \langle \mathbf{G}_1 \rangle, \dots, \langle \mathbf{G}_t \rangle) \end{aligned}$$

symbolic
method

transfers to a system of GF equations

$$\begin{aligned} G_0(z) &= F_0(G_0(z), G_1(z), \dots, G_t(z)) \\ G_1(z) &= F_1(G_0(z), G_1(z), \dots, G_t(z)) \\ &\dots \\ G_t(z) &= F_t(G_0(z), G_1(z), \dots, G_t(z)) \end{aligned}$$

Grobner basis
elimination

that reduces to a *single* GF equation

$$G_0(z) = F(G_0(z), G_1(z), \dots, G_t(z))$$

Drmota-Lalley-Woods
theorem

that has an *explicit* solution

$$G(z) \sim c - a\sqrt{1 - bz}$$

singularity analysis

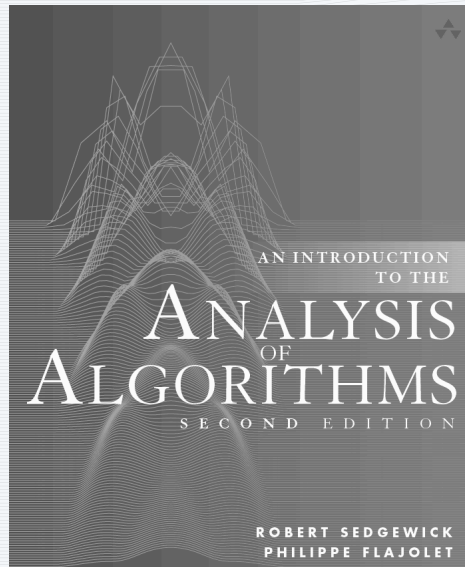
that transfers to a
simple asymptotic form

$$G_N \sim \frac{a}{2\sqrt{\pi N^3}} b^N \quad !!$$

Stay tuned for many more (in Part 2).

ANALYTIC COMBINATORICS

PART ONE



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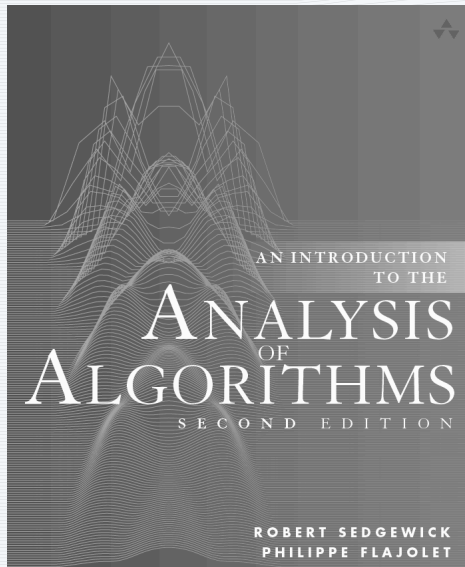
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- **Coefficient asymptotics**
- Perspective

5c.AC.Asymptotics

ANALYTIC COMBINATORICS

PART ONE



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5. Analytic Combinatorics

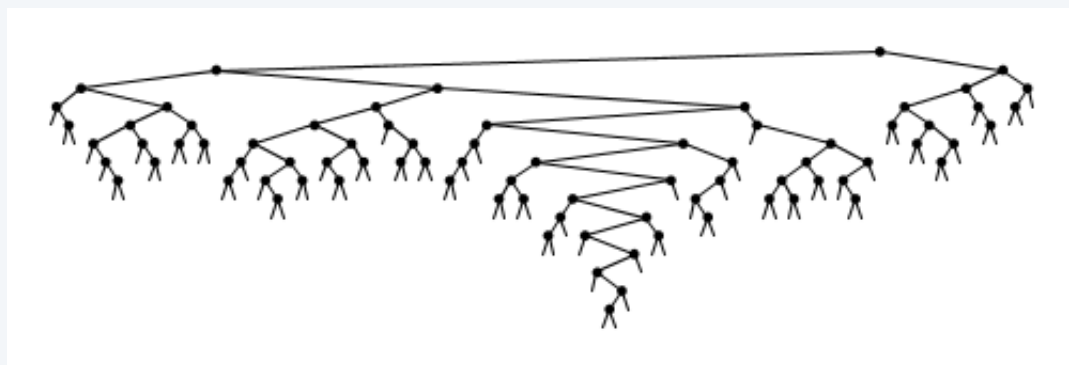
- The symbolic method
- Labelled objects
- Coefficient asymptotics
- **Perspective**

5d.AC.Perspective

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = \frac{1}{2z} (1 - \sqrt{1 - 4z})$$

GF

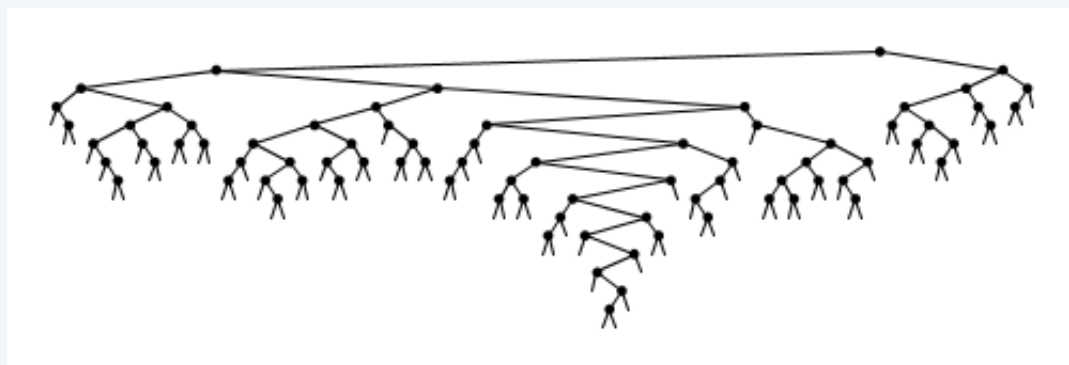
$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient
asymptotics

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many binary trees with N nodes?



$$T = E + Z \times T \times T$$

combinatorial construction

$$T(z) = 1 + zT(z)^2$$

GF equation

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

coefficient
asymptotics

Note: With complex asymptotics, we can transfer directly from GF equation (no need to solve it). See Part 2.

Old vs. New: Two ways to count binary trees

Old

Recurrence → GF

Expand GF

Asymptotics

Solving the Catalan recurrence with GFs

Recurrence that holds for all N .

$$T_N = \sum_{0 \leq k < N} T_k T_{N-1-k} + \delta_N$$

Multiply by z^N and sum.

$$T(z) \equiv \sum_{N \geq 0} T_N z^N = \sum_{N \geq 0} \sum_{0 \leq k < N} T_k T_{N-1-k} z^N$$

Switch order of summation

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N > k} T_k T_{N-1-k} z^N$$

Change N to $N+k+1$

$$T(z) = 1 + \sum_{k \geq 0} \sum_{N \geq 0} T_k T_N z^{N+k+1}$$

Distribute.

$$T(z) = 1 + z \left(\sum_{k \geq 0} T_k z^k \right) \left(\sum_{N \geq 0} T_N z^N \right)$$

$$T(z) = 1 + zT(z)^2$$

Solving the Catalan recurrence with GFs (continued)

Functional GF equation.

$$T(z) = 1 + zT(z)^2$$

Solve with quadratic formula.

$$zT(z) = \frac{1}{2} \left(1 \pm \sqrt{1 - 4z} \right)$$

Expand via binomial theorem.

$$zT(z) = -\frac{1}{2} \sum_{n \geq 1} \binom{2n-2}{n-1} (-4z)^n$$

Set coefficients equal

$$T_N = -\frac{1}{2} \sum_{n \geq 1} \binom{2n-2}{n-1} (-4)^n z^{n-1}$$

Expand via definition.

$$= -\frac{1}{2} \sum_{n \geq 1} \frac{(2n-2)!}{(n-1)! (n-1)!} (-4)^n z^{n-1}$$

Distribute $(-2)^N$ among factors.

$$= \frac{1}{2} \sum_{n \geq 1} \frac{(2n-2)!}{(n-1)! (n-1)!} 2^n z^{n-1}$$

Substitute $(2/1)(4/2)(6/3)\dots$ for 2^n .

$$= \frac{1}{2} \sum_{n \geq 1} \frac{(2n-2)!}{(n-1)! (n-1)!} 2^n z^{n-1}$$

$$= \frac{1}{2} \sum_{n \geq 1} \frac{(2n-2)!}{(n-1)! (n-1)!} 2^n z^{n-1}$$

Inclass exercise

Given Stirling's approximation $\ln N! = N \ln N - N + \ln \sqrt{2\pi N} + O(1/N)$

Develop an asymptotic approximation for $\binom{2N}{N}$ to $O(1/N)$ (relative error)

$$\begin{aligned} \binom{2N}{N} &= \exp(\ln(2N!) - 2 \ln N!) \\ &= \exp(2N \ln(2N) - 2N + \ln \sqrt{4\pi N} + O(1/N) \\ &\quad - 2(N \ln N - N + \ln \sqrt{2\pi N} + O(1/N))) \\ &= \exp(2N \ln 2 - \ln \sqrt{\pi N} + O(1/N)) \\ &= \frac{4^N}{\sqrt{\pi N}} (1 + O(1/N)) \end{aligned}$$

$$\ln \sqrt{4\pi N} - 2 \ln \sqrt{2\pi N} = \ln 2 - 2 \ln \sqrt{2} - \ln \sqrt{\pi N} = -\ln \sqrt{\pi N}$$

$$\text{Ex. } \frac{1}{4^N} \binom{2N}{N} \sim \frac{1}{\sqrt{\pi N}}$$

26

New

$$T = E + Z \times T \times T$$

$$T(z) = 1 + zT(z)^2$$

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Ex: How many generalized derangements?



$$D_M = SET(CYC_{>M}(Z))$$

combinatorial construction

$$\frac{e^{-z} - z^2/2 \dots - z^M/M}{1 - z}$$

GF equation

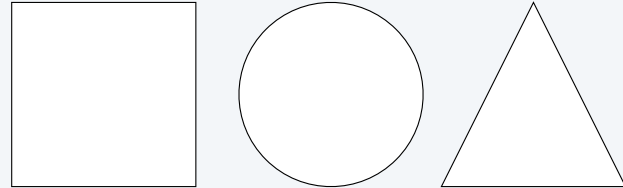
$$\sim \frac{N!}{e^{H_M}}$$

coefficient
asymptotics

A standard paradigm for analytic combinatorics

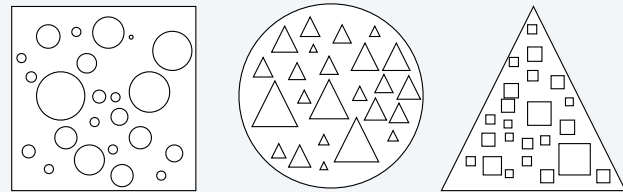
Fundamental constructs

- elementary or trivial
- confirm intuition



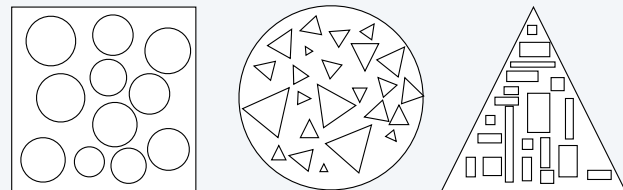
Compound constructs

- many possibilities
- classical combinatorial objects
- expose underlying structure



Variations

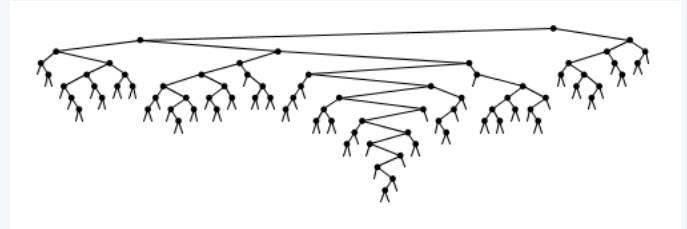
- unlimited possibilities
- *not* easily analyzed otherwise



Combinatorial parameters

are handled as two counting problems via cumulated costs.

Ex: How many leaves in a random binary tree?



1. Count trees

$$T = E + Z \times T \times T$$

$$T(z) = \frac{1}{2z} (1 - \sqrt{1 - 4z})$$

$$T_N \sim \frac{4^N}{\sqrt{\pi N^3}}$$

2. Count leaves in all trees

$$T = E + Z \times T \times T$$

$$T_u(1, z) = \frac{z}{\sqrt{1 - 4z}}$$

$$C_N \sim \frac{4^{N-1}}{\sqrt{\pi N}}$$

Symbolic method works for BGFs (see text)

3. Divide

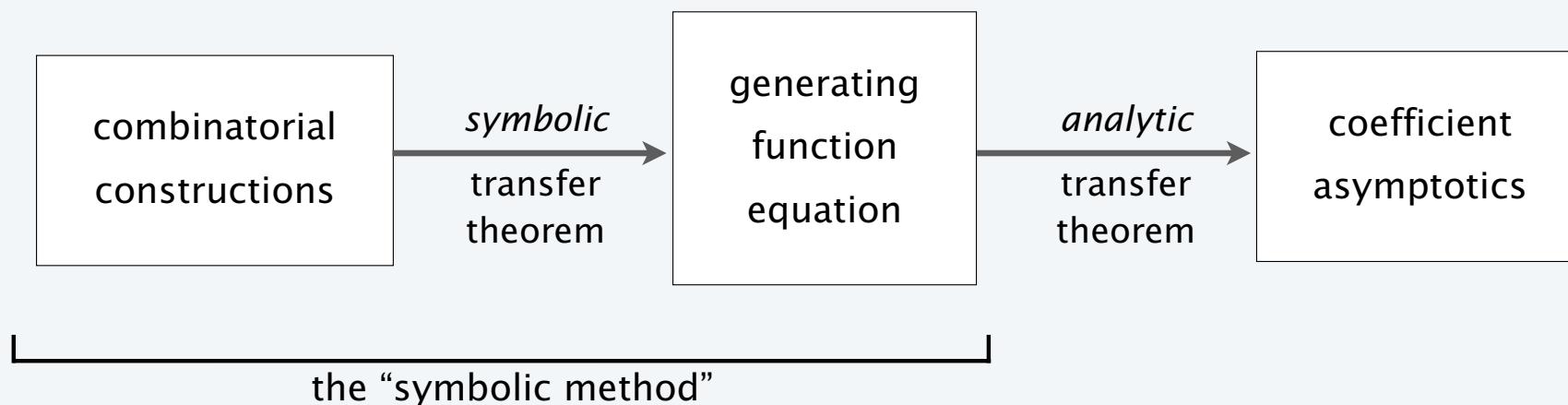
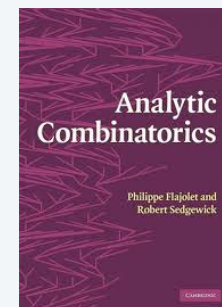
$$\frac{C_N}{T_N} \sim \frac{N}{4}$$

Analytic combinatorics

is a calculus for the quantitative study of large combinatorial structures.

Features:

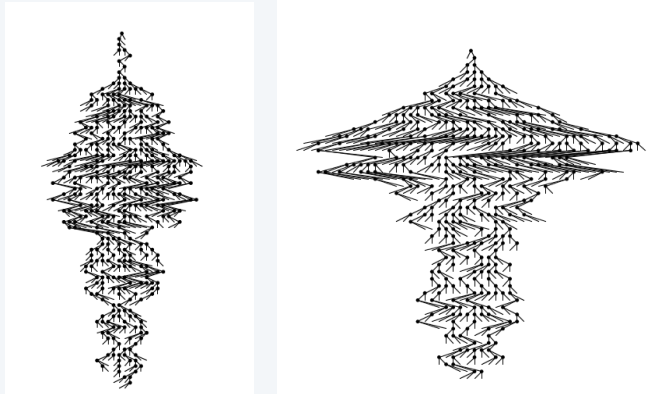
- Analysis begins with formal *combinatorial constructions*.
- The *generating function* is the central object of study.
- *Transfer theorems* can immediately provide results from formal descriptions.
- Results extend, in principle, to any desired precision on the standard scale.
- Variations on fundamental constructions are easily handled.



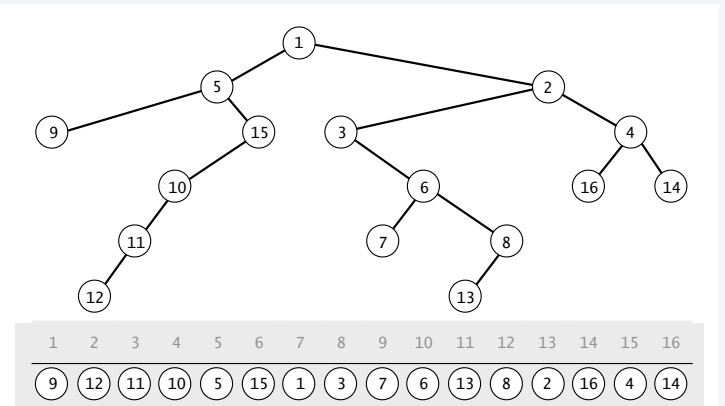
Stay tuned

for many applications of analytic combinatorics *and applications to the analysis of algorithms*

Trees



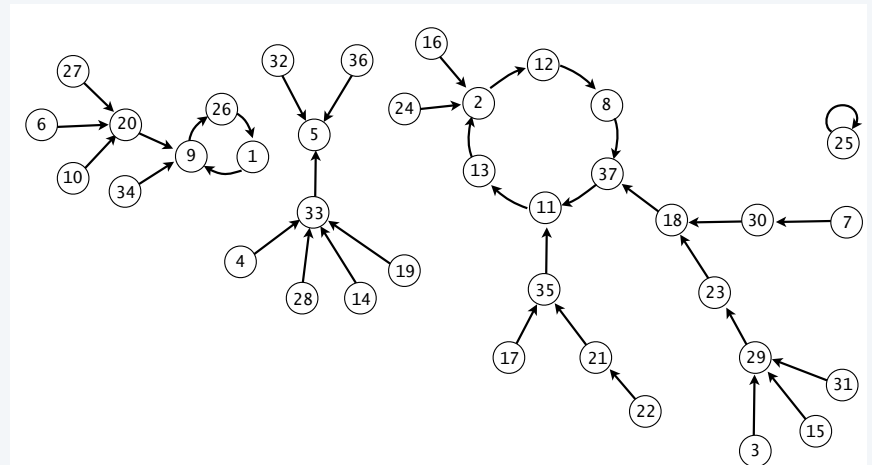
Permutations



Mappings

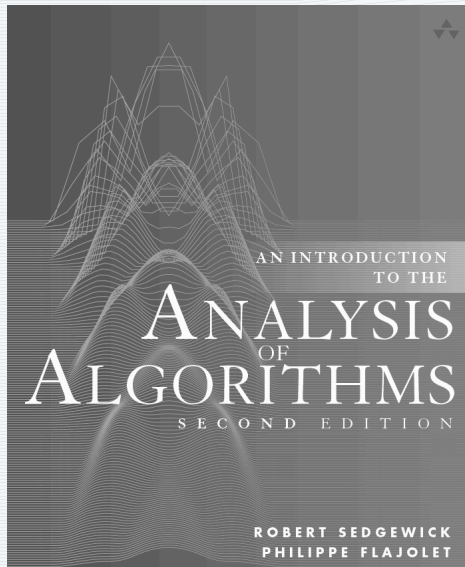
Bitstrings

10111110100101001100111000100111110110110100000111100001100111011101111101011000
1101001010001111101001110011010011101010111110000010110111001101000000111001110
1110111010100111010111001101000110000110011001100111110011001000011001000101010010
101110000110110001100111011100110110111110011011000011001100101000000110
101011001110100011011011101100100101010010101111100110000001111101000001111
1000001001100000110001100010000111100111001111000001100111110011011000100100111
1000101011100011101011000001110101010001011000100110111110011110110010
0011101100101110010001100001001111010010011001100001100111010011010000101000111
001111110011011011101101101010011011100011111111010111010011000000100101110
1010100011110000101000001100100000110101001010001100110010101011101101111110
110000001011101101100010101101100100100000110111001000000110101000000101000
1110111011011110111111101001110100101111101101001110100110001100010010010
00111111100111011011111000100011100001110101110010101111100111010101111



ANALYTIC COMBINATORICS

PART ONE



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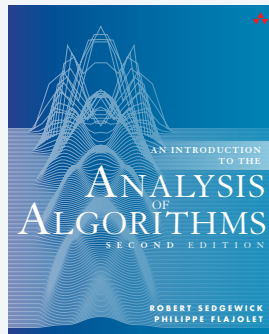
5. Analytic Combinatorics

- The symbolic method
- Labelled objects
- Coefficient asymptotics
- **Perspective**

5d.AC.Perspective

Exercise 5.1

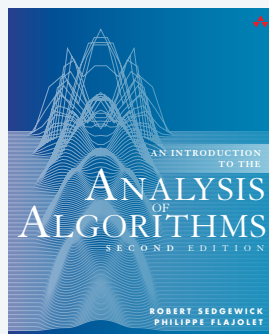
Practice with counting bitstrings.



Exercise 5.1 How many bitstrings of length N have no 000?

Exercise 5.3

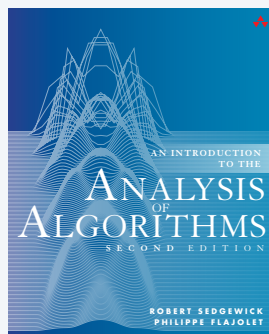
Practice with counting trees.



Exercise 5.3 Let $\mathcal{U}$ be the set of binary trees with the size of a tree defined to be the total number of nodes (internal plus external), so that the generating function for its counting sequence is $U(z) = z + z^3 + 2z^5 + 5z^7 + 14z^9 + \dots$. Derive an explicit expression for $U(z)$.

Exercise 5.7

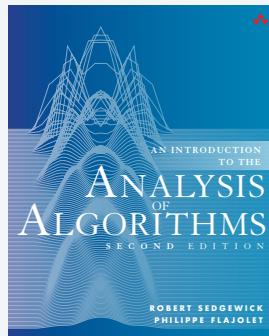
Practice with counting permutations.



Exercise 5.7 Derive an EGF for the number of permutations whose cycles are all of odd length.

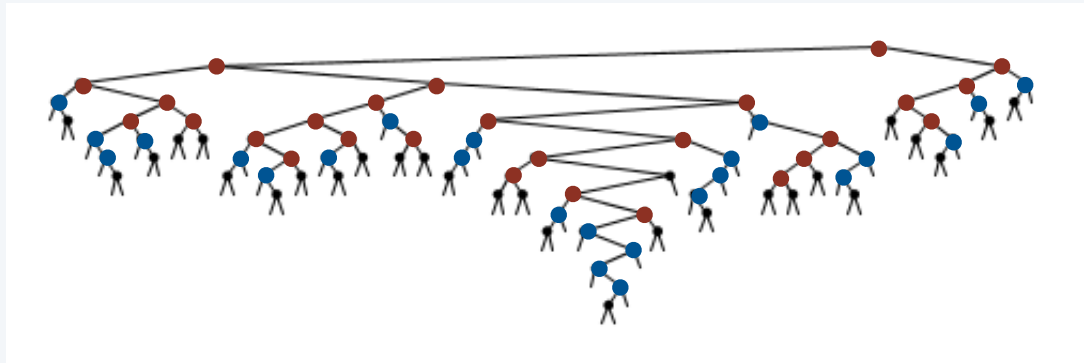
Exercises 5.15 and 5.16

Practice with tree parameters.



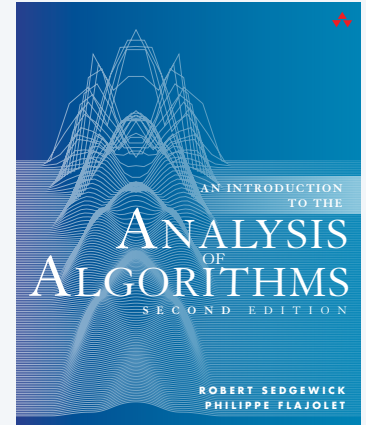
Exercise 5.15 Find the average number of internal nodes in a binary tree of size n with both children internal. ●

Exercise 5.16 Find the average number of internal nodes in a binary tree of size n with one child internal and one child external. ●



Assignments for next lecture

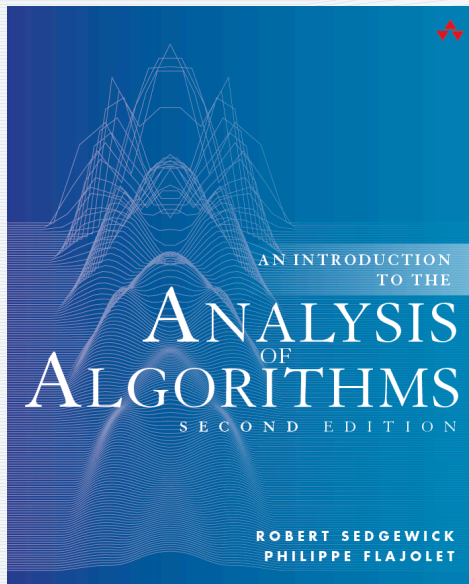
1. Read pages 219-255 in text.



2. Write up solutions to Exercises 5.1, 5.3, 5.7, 5.15, and 5.16.

ANALYTIC COMBINATORICS

PART ONE



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5. Analytic Combinatorics